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REVIEW ON X-FEM FOR FRACTURE MECHANICS: CURRENT DEVELOPMENTS AND FUTURE PROSPECTS

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***Abstract**- The Extended Finite Element Method (X-FEM) is a numerical method that is designed for treating many problems characterized by discontinuities, localized deformations, singularities and complex geometries in material modeling. The idea behind X-FEM is to retain mesh free methods while alleviating the deficiency of traditional finite element method (FEM). In fracture mechanics, it extends the classical FEM approach by enriching the approximation space so that it is able to naturally reproduce the challenging feature associated with the discontinuous functions. Based on a standard Galerkin procedure and the use of partition of unity concept, the X-FEM is used to model internal (or external) boundaries without the mesh to conform to these boundaries in the discrete model. This paper surveys the recent and future potential of X-FEM for computational fracture mechanics. Here we highlighted the new developments, improvements and applications of X-FEM to analyse and study the complex material fracture mechanisms for modeling the interfaces with discontinuities.*

Keywords: Extended finite element method; Fracture mechanics, Material science, Material modeling; Level Set Method (LSM)

1. INTRODUCTION

The X-FEM and the FEM are versatile tools for the simplification of the modeling of discontinuous phenomena in material science. In conventional Finite Element framework, it is difficult to model the crack growth due to the topology alteration of the mesh. During the modeling of evolving discontinuities the mesh must be regenerated at each step and the crack tips must be placed accurately to allow the material separation along the crack surface [1]. In X-FEM continuous refinement of mesh to conform the discontinuities is not required. The internal boundaries in the discrete X-FEM model is accommodate by the concept partition of unity [2, 3] and based on a standard Galerkin procedure. This was proposed by Belytschko and Black [4]. They enriched the finite element approximation with additional basis functions to introduce a discontinuous displacement field along with the crack surface in X-FEM. The asymptotic near tip field and a Heaviside function H(x) was introduced by Dolbow et al. [5,6]and Moes et al. [7] to improve the technique and Sukumar et al.[8] extended the concept for three dimensional static crack modeling. The arbitrary discontinuities and discontinuous derivatives in finite elements were combined with modeling functions by Belytschko et al. [9].In contrast, the accuracy of the element enrichment schemes of Benzley [10] based on a partition of unity, is uninfluenced by the element size for a large range. Moreover, the technique becomes weak as the transition elements are required for crack tip elements and the decrease of element size near the crack tip elements. The only drawback of this method is the need for a variable number of degrees of freedom per node.

The X-FEM unified with level set method (LSM) for modeling the entire fracture such as the geometry and the displacement field of crack and other engineering problems in the complex domains can easily be constructed where FEM faces difficulties to produce accurate solution.

2. GENERAL REVIEW

2.1 Multiple cracks and crack nucleation

Daux et al. [11] presented a methodology to model the voids and complex geometries like cracks with multiple branches by adapting X-FEM. This study facilitates to model the cracks associated with holes in a system. For multiple cracks evolution Budyn et al. [12] studied a modeling method with the X-FEM. Mariano and Stazi[13] applied the X-FEM to a multi-field model of micro-cracked bodies to describe the relation between a macro-crack and a number of micro-cracks.

For modeling crack nucleation Bellec and Dolbow [14] focused on the particular case where the extent of the crack approaches the support size of the nodal shape functions. For the resolution of complex crack patterns Remmers et al. [15] analyzed the prospect of defining interrelated segments at arbitrary locations and in arbitrary directions. There by it will allow taking decision about the crack patterns including crack nucleation and other complex cracks, which are observed during the growth and coalescence.

2.2 Holes and Inclusion

Sukumar et al. [16] exposed a method to model arbitrary holes and material interfaces without the need of massing internal boundaries in two-dimensional linear elastostatics. For the treatment of holes, material inclusions and cracks Legrain et al. [17] analyzed the stability of incompressible formulations unified with X-FEM. Sukumar and al.[18] applied the X-FEM to understand the toughening mechanisms in polycrystalline materials such as ceramics. They focused on the micro-structural effects in the brittle fracture to construct a two-dimensional numerical model.

2.3 Graded Materials

Dolbow and Nadeau [19] raised some essential issues pertaining to the application of effective properties for the failure analysis in the micro-structured materials. These fundamental theoretical and numerical issues are concerned with the functionally graded materials FGMs. Numerous methods for the computation of various stress intensity factors in functionally graded materials FGMs are described. A new interaction energy integral method was presented by Dolbow and Gosz [20] for the calculation of mixed-mode stress intensity factors at the tips of arbitrarily oriented FGMs.For the simplified calculation of the various stress intensity factors in FGMs Menouillard et al. [21] presented a general method to solve that. Comi and Stefano Mariani [22] developed an extended finite element simulation of quasi-brittle fracture and an ad hoc formulation. The exposed formula was utilized to track the crack propagation in the graded medium.

2.4 Material interfaces

Belytschko [23] proposed a simplified method with implicit functions to define a solid object having material interfaces, sliding surfaces and cracks on the outside surface and in any inner surfaces by structured finite elements. In the two-dimensional elastostatic system Nagashima et al. [24] studied the bi-material interface cracks problem. They applied asymptotic solution of a homogeneous (not interface) crack to enrich the crack tip nodes, and adopted a fourth order Gauss integration for a 4-node isoparametric element with enriched nodes.Liu et al.[25] improved the accuracy of the crack-tip displacement field and determined mixed mode stress intensity factors (SIFs) directly by taking into account higher order terms of the asymptotic crack-tip displacement field with the help of X-FEM for homogeneous materials as well as for bi-materials. Sukumar et al.[26] proposed a partition of unity enrichment techniques for bi-material interface cracks. The functions for the crack-tip enrichment are selected by the span of asymptotic displacement fields for an interfacial crack. The stress intensity factors were numerically determined by the domain form of the interaction integral in the bi-material interfacial cracks. Hettich and Ramm [27] extend the X-FEM to constructed a mechanical modeling of material interfaces and interfacial cracks.

Asadpoure et al. [28,29], and Asadpoure and Mohammadi [30] developed enrichment functions for X-FEM analysis of crack in orthotropic media. They proposed three independent sets of orthotropic enrichment functions in the three planes/axes of symmetry. Further, Piva et al. [31] studied the elastodynamic problems with the orthotropic crack tip solutions. Yan and Park [32] elucidated a study for the simulation of crack growth in layered composite structures by the X-FEM. Through the study they analyzed the capability of X-FEM to predict the crack path in near-interfacial fracture.

2.5 Crack cohesive

To investigate fracture of concrete materials Wells and Sluys [33] unified the X-FEM with the cohesive zone model. Compared with the experiment they found excellent mixed-mode crack prediction. Moes and Belyschco [34] applied the X-FEM to predict the crack growth and its performance where a cohesive law was involved on the crack faces. Zi and Belyschco [35] exposed a new enrichment technique for the treatment of curved cracks with higher order enrichments. In quasi-brittle materials Mariani and Perego [36] proposed a simulation technique to model the quasi-static cohesive bi-dimensional crack propagation.

Xiao et al. [37] proposed an incremental-secant modulus iteration scheme and stress recovery described by cohesive crack models to simulate the cracking process in quasi-brittle materials. The softening law of the cohesive crack models is composed of linear segments. In brittle and quasi-brittle solids Meschke and Dumstor [38] elucidated a variational X-FEM to track the cohesion less and cohesive cracks.

2.6 Contact and Friction

Dolbow et al. [40] exposed three different interfacial constitutive laws as a technique for the finite element modeling of fracture in 2D crack growth with frictional contact. The developed laws are perfect contact and unilateral contact with or without friction. Vitali and Benson [41] demonstrated that the X-FEM for contact in Multi-Material Arbitrary Lagrangian–Eulerian (MMALE) formulations produce better results than the technique with the mixture theories. And their technique also agree with the Lagrangian solution. Vitali and Benson [42] extend their investigation to friction and enriched the accuracy of their previous study. Khoei and Nikbakht [43,44] enriched the classical finite element approximation by applying additional terms to simulate the frictional behavior of contact between two bodies. Ribeaucourt et al. [45] applied X-FEM /LATIN method to simulate the fatigue frictional contact crack propagation.

Borja [46] described the assumed enhanced strain (AES) and the X-FEM for simulating the frictional crack propagation. For the frictional contacts Boucard et al. [47] exposed a multi scale strategy for the structural optimization of geometric discontinuities. In [48] they proposed a new crack tip element for the phantom-node method with arbitrary cohesive cracks. Liu and Borja [49] applied X-FEM to present a Standard Coulomb plasticity model for predicting the frictional crack propagation on the surface of discontinuity.

3. X-FEM CONCEPTS

3.1. Crack-tip enrichment

The X-FEM and the generalized FEM [50–53] are closely related to each other as they both belonging to the class of partition of unity. In X-FEM the fundamental characteristics is that the addition of discontinuous enrichment functions by using the partition of unity to the finite element approximation.

$$u(x) = \sum_{i=1}^{n} N_i(x)\left(u_i + \sum_{j=1}^{ne(i)} a_{ji} F_j(r,\theta)\right), \qquad (1)$$

Where polar coordinate system is ($r,\ \theta$) at the crack tip$N_i(x)$are the standard finite element shape functions. a_{ji} is enrichment coefficient related to nodes and the number of coefficients *ne(i)* is for node I. *ne(i)* is four all the nodes around crack tips and zero at all other nodes. F_j ($r,\ \theta$) are the crack-tip enrichment functions in isotropic elasticity and they are found from the asymptotic displacement fields:

$$\{Fj(r,\theta)\}_{j=1}^{4} = \left\{\sqrt{r}\sin\left(\frac{\theta}{2}\right); \sqrt{r}\cos\left(\frac{\theta}{2}\right), \sqrt{r}Sin\left(\frac{\theta}{2}\right)\sin\theta; \sqrt{r}\cos\left(\frac{\theta}{2}\right)\sin\theta\right\} \qquad (2)$$

In the above equation the first function is discontinuous across the crack and represents the discontinuity near the tip. The remaining three functions in the equation are for assessing accurate results with relatively coarse meshes (Fig. 1).

3.2 Heaviside function

The Heaviside jump function is a discontinuous function across the crack surface. It have constant value for each side of the crack and they are +1 on one side and -1 on the other. After unified with the jump function, the final approximation will be changed to the following formula

$$U = \sum_{i\in I} \mathrm{u}_i N_i + \sum_{j\in J} b_j N_j H(x) + \sum_{k\in K1} N_k\left(\sum_{l=1}^{4} C_k^{l1} F_l^1(x)\right) + \sum_{k\in K2} N_k\left(\sum_{l=1}^{4} C_k^{l2} F_l^2(x)\right) \qquad (3)$$

Where, $N_i$is the shape function for node *i*, *I* is the set of all nodes of the domain, *J* is the set of nodes whose shape function support is cut by a crack, *K* is the set of nodes whose shape function support with crack front,$u_i$are the classical degrees of freedom or displacement for node *i*,$b_j$account for the jump in the displacement field across the crack at node *j*. It represents opening of the crack if the crack is aligned with the mesh, *H(x)* is the Heaviside function, C_k^l are the additional degrees of freedom associated with the crack-tip enrichment functions F_1,F_1 is an enrichment which corresponds to the four asymptotic functions in the development expansion of the crack-tip displacement field in a linear elastic solid (Fig. 2).

3.3 Numerical integration

The numerical integration of X-FEM function faces two complications: the singularity at the crack tip and the discontinuity along the crack. The cut elements are generally integrated by partitioning them into standard sub-elements. Gauss quadrature rule must be applied for better results in the sub-elements. A set of sub-triangles was formed by dividing each side of a cut element into triangles in the earlier investigations. Some authors implemented slightly different approach to form a set of sub-quadrilaterals by dividing the cut elements (Fig. 3).

The numerical integration procedure to partition cut elements by the crack is as follows:

1. Partition of cut elements to form the Delaunay triangulation to get the sub-elements.
2. The coordinates and weights of Gauss points are computed and then converted into the parent coordinate system for each sub-elements of the original element.

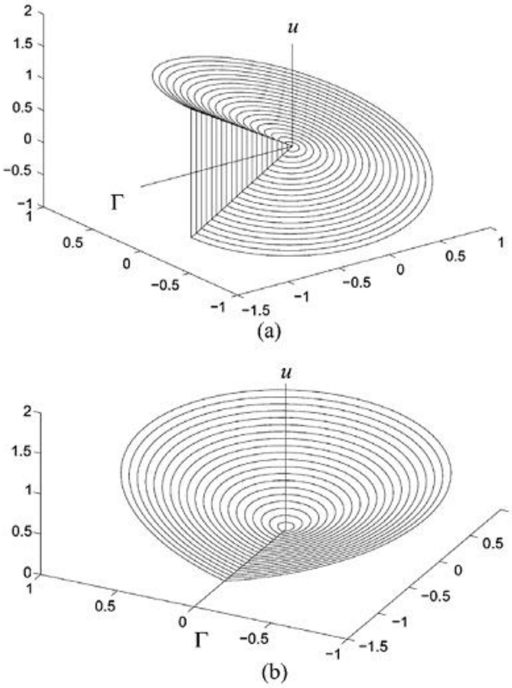

Fig.1: Branch functions for discontinuities in: (a) the function; and (b) its derivative [9].

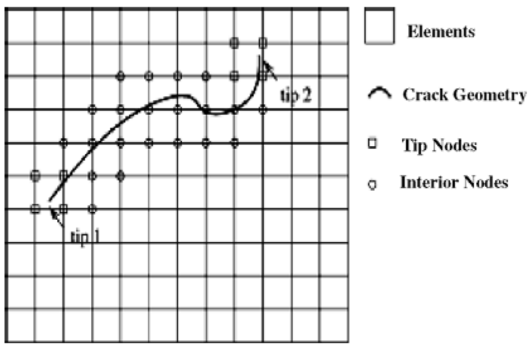

Fig.2: A strategy of enrichment [19].

Ventura [54] discussed the eradication of quadrature sub cells for discontinuity functions in the X-FEM. During the investigation, he exposed the application of standard Gauss quadrature in the elements with the discontinuity without splitting the elements in sub-cells or to presenting any additional approximation.

3.4 Level set method

The level set method (LSM) is a conceptual framework for using level sets as a tool for modeling the motion of interfaces. This numerical scheme is developed by Osher and Sethian [55].The principle of the method is to numerical analysis of an interface by the level set function (zero of a function) and Hamilton–Jacobi equations,where the function is updated with the equation considering the speed of the interface in the direction normal to this interface.

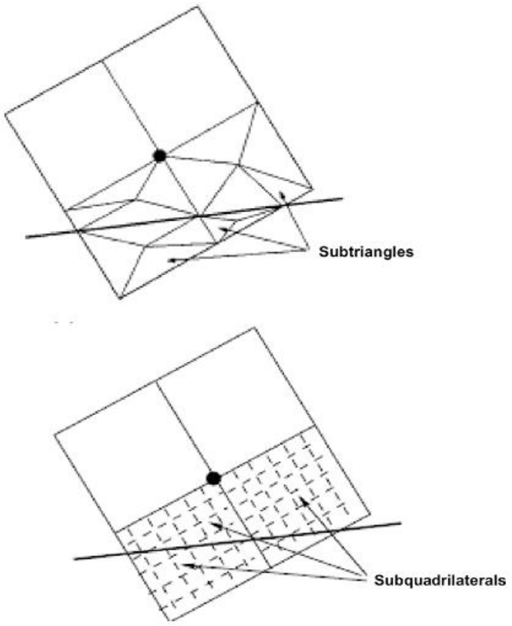

Fig.3: Partitioning into standard sub-elements [19].

The significant advantage of coupling LSM with the X-FEM is that the LSM makes it very easy to locate crack and crack tips location. For the growth of the crack, X-FEM compute the stress and displacement fields. Stolarska et al.[56] presented the first algorithm coupling the LSM with X-FEM to solve the crack growth in two dimensions. A crack is described by two level sets:

(1) a normal level set, $\psi(x)$, which the signed distance to the crack surface,
(2) a tangent level set $\phi(x)$, which is the signed distance to the plane including the crack front and perpendicular to the crack surface.

In a given element,ψ_{min} andψ_{max}, respectively, be the minimum and maximum nodal values of ψ on the nodes of that element. Similarly, let, ϕ_{min} and ϕ_{max} , respectively be the minimum and maximum nodal values of ϕ on the nodes of an element:

–If $\phi< 0$ and, $\psi_{min}\psi_{max}\leq 0$, then the crack cuts through the element and the nodes of the element are to be enriched with *H(x).*

– If in that element$\phi_{min}\phi_{max}\leq 0$ and$\psi_{min}\psi_{max}\leq 0$, then the tip lies within that element, and its nodes are to be enriched *Fi(r,θ).*

At point *x*, Stolarska et al. [56] presented the radius from the crack tip and the angle of deviation from the tangent to the crack tip:

$$r=\sqrt{\psi^2(x,t)+\phi^2(x,t)} \text{ And } \theta=\tan^{-1}\frac{\psi(x,t)}{\phi(x,t)} \qquad (4)$$

A particular level set method Fast Marching Method (FMM) [57] in association with X-FEM was also applied to simulate planar three-dimensional fatigue of crack growth. This coupling was applied by Sukumar et al.[58] for modeling simple crack growth andChopp and Sukumar [59] have studied the fatigue crack propagation of multiple coplanar cracks. In [60], Sukumar et al. presented the non-planar three-dimensional crack growth simulations by a unified process of X-FEM and FMM.

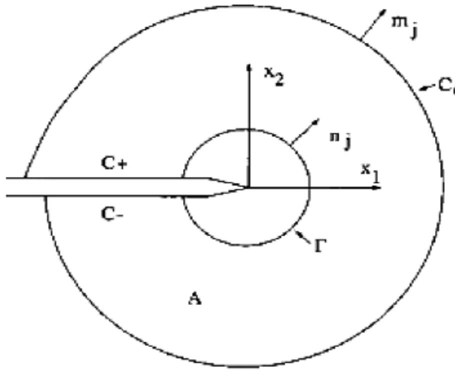

Fig.4: Definition of the*J* integral around a crack.

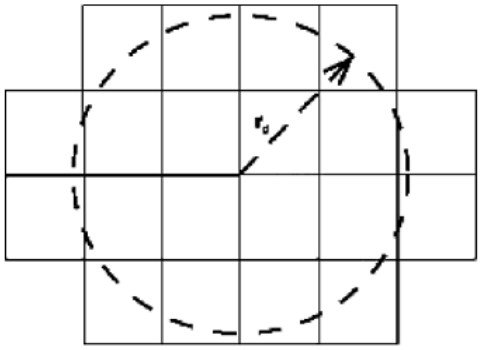

Fig.5: Elements selected about the crack tip for calculation of the interaction integral [7]

3.6Interaction integral

The domain forms of the interaction integrals are employed to calculate the stress intensity factors. According to figure 4, the J-integral can be defined as:

$$I^{(1,2)}=\int_\Gamma\left[W^{(1,2)}\delta_{1j}-\sigma_{ij}^{(1)}\frac{\partial u_i^{(2)}}{\partial_{x1}}-\sigma_{ij}^{(2)}\frac{\partial u_i^{(1)}}{\partial_{x1}}\right]njd\Gamma \qquad (5)$$

The contour integral in equation (5) is not in a suitable form for finite element calculations. Moes et al. [7] alter the integral by multiplying the integrand by a sufficiently smooth weighting function *q(x)* into an equivalent domain form. The weighting function *q(x)* on an open set with crack tip containing the value of unity and vanishes on an outer prescribed contourC_O.

$$I^{(1,2)}=\int_A\left[\sigma_{ij}^{(1)}\frac{\partial u_i^{(2)}}{\partial x1}+\sigma_{ij}^{(2)}\frac{\partial u_i^{(1)}}{\partial x1}-W^{(1,2)}\delta_{1j}\right]\frac{\partial q}{\partial x_j}dA \qquad (6)$$

In the equation, the domain A is set from the collection of elements about the crack tip. The quantity h_{local} is designated as the characteristic length of an element touched by the crack tip. This quantity is calculated by the square root of the element area for two-dimensional analysis. Then the domain A contains all elements which have a node within a ball of radius r_d about the crack tip (Figs. 5 and 6).

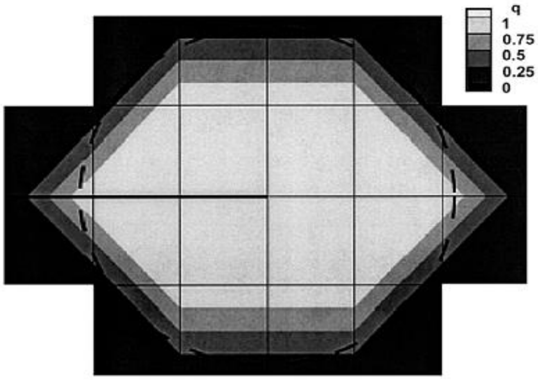

Fig.6. Weight function q on the elements [7].

4. CONCLUSION

The major concepts and an overall recent progress of the X-FEM in the crack growth modeling analysis have been reviewed in this article. It exposes the essential stages and numerical modeling processes carried out by the finite community element with respect to the fracture mechanics. To simulate and analysis the complex material fracture mechanics, the review points out the potentiality of the X-FEM. The method makes it possible to predict the crack growth independently of the mesh in a cracked domain. This is the major advantage of X-FEM over the conventional methods where remeshing and interpolation is necessary at the each stage of crack computation, which can be sources of instabilities.

When X-FEM is combine with level sets, the entire representation of the feature in the complex domains, can be constructed accurately in terms of nodal values at the nodes of the original mesh, which is difficult to solve using standard FEM. Furthermore, user can benefit from the many built-in features of such code used in the sub-structuring approach X-FEM/FEM software.

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